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ON THE SOLUTION OF THE n-DIMENSIONAL OPERATOR RELATED TO THE DIAMOND OPERATOR

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Abstract

In this paper, we consider the solution of the equation $\lozenge_c^k u(x) = \sum_{r=0}^m C_r \lozenge_c^r \delta,$ where $\lozenge_c^k$ is the operator related to the diamond operator iterated k-times and is defined by

$$\lozenge_c^k = \left[\frac{1}{c^4}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^2\right]^k.$$

Now $x \in \mathbb{R}^n$ is the n-dimensional Euclidean space, $p + q = n$, C_r is a constant, δ is the Dirac-delta distribution and $\lozenge_c^0 \delta = \delta$ and $k = 0, 1, 2, \ldots.$

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It is found that the type of solution of this equation, such as the ordinary function, the tempered distributions and the singular distributions depend on the relationship between the values of k and m.

1. Introduction

Kananthai [4] showed that the solution of the convolution form $u(x) = R_{2k,c_1}(x) * R_{2k,c_2}(x)$ is a unique elementary solution of the equation $\Box_{c_1}^k \Box_{c_2}^k u(x) = \delta$, where $\Box_{c_1}^k$ and $\Box_{c_2}^k$ are the operators which related to the ultra-hyperbolic type operators iterated k-times and δ is the Dirac-delta distribution and in particular, if $k = p = 1$ with $x_1 = t$ (times), c_1 and c_2 are velocities, then $u(x) = R_{2,c_1}(x) * R_{2,c_2}(x)$ is the elementary solution of the elastic wave equation of fourth order. Sritanratana and Kananthai [6] studied the product of the nonlinear diamond operators related to the elastic wave and also introduced the ultra-hyperbolic operator $\Box_c^k$. Consider the operator related to the ultra-hyperbolic operator iterated k-times defined by

$$\Box_c^k = \left[\frac{1}{c^2} \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k.$$

Trione [8] showed that the generalized function $R_{2k,1}(x)$ defined by (2.2) is the unique elementary solution of the operator $\Box_1^k$, that is, $\Box_1^k R_{2k,1}(x) = \delta$, where $x \in \mathbb{R}^n$ is the n-dimensional Euclidian space. Also, Tellez [7, pp. 147-149] proved that $R_{2k,1}(x)$ exists only if n is an odd with p odd and q even or only n is an even with p odd and q odd. Moreover, Bupasiri and Nonlaopon [1] studied the weak solution of compound equations related to the ultra-hyperbolic operators of the form

$$\sum_{r=0}^{m} C_r \Box_c^r u(x) = f(x).$$

Furthermore, we also know that the function $E(x)$ defined by (2.4) is an elementary solution of the operator related to the Laplace operator

$$\Delta_c^k = \left[\frac{1}{c^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k,$$

that is, $\Delta_c E(x) = \delta$, where $x \in \mathbb{R}^n$.

Now, in this paper, the operator related to the diamond operator can be expressed as the product of the operator $\square_c$ and Δ_c, that is,

$$\begin{aligned}\Diamond_c^k &= \left[\frac{1}{c^4}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \\ &= \left[\frac{1}{c^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k\left[\frac{1}{c^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k \\ &= \square_c^k\Delta_c^k. \end{aligned} \tag{1.1}$$

Now we are finding the solution of the equation

$$\Diamond_c^k u(x) = \sum_{r=0}^{m} C_r \Diamond_c^r \delta$$

or

$$\square_c^k\Delta_c^k u(x) = \sum_{r=0}^{m} C_r \square_c^k\Delta_c^k \delta. \tag{1.2}$$

In finding the solutions of (1.2), we use the method of convolutions of the generalize functions. Before going to that point, the following definitions and some concepts are the needs.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, ..., x_n)$ be a point of the *n*-dimensional space $\mathbb{R}^n$,

$$V = c^2(x_1^2 + x_2^2 + \cdots + x_p^2) - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{2.1}$$

where $p + q = n$. Then define $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } V > 0\}$ which designates the interior of the forward cone and $\overline{\Gamma}_+$ designates its closure and the following functions introduce by Nozaki [5, p. 72] that

$$R_{\alpha, c}(x) = \begin{cases} \dfrac{V^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+, \\ 0, & \text{if } x \notin \Gamma_+, \end{cases} \tag{2.2}$$

$R_{\alpha,1}(x)$ is called the *ultra-hyperbolic kernel of Marcel Riesz*. Here α is a complex parameter and *n* is the dimension of the space. The constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{2+\alpha-n}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right)\Gamma\left(\frac{p-\alpha}{2}\right)} \tag{2.3}$$

and *p* is the number of positive terms of

$$V = c^2(x_1^2 + x_2^2 + \cdots + x_p^2) - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \quad p + q = n$$

and let $\operatorname{supp} R_{\alpha, c}(x) \subset \overline{\Gamma}_+$. Now $R_{\alpha, c}(x)$ is an ordinary function if $\operatorname{Re}(\alpha, c) \geq n$ and is a distribution of α if $\operatorname{Re}(\alpha, c) < n$.

Definition 2.2. Let

$$x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$$

and

$$|x| = \sqrt{c^2(x_1^2 + x_2^2 + \cdots + x_p^2) + x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2}$$

and let the function $E(x)$ be defined by

$$E(x) = \frac{|x|^{2-n}}{(2-n)w_n}, \tag{2.4}$$

where

$$w_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

is a surface area of the unit sphere. Let the function

$$S_{\alpha,c}(x) = 2^{-\alpha}\pi^{-\frac{n}{2}}\Gamma\left(\frac{n-\alpha}{2}\right)\frac{|x|^{\alpha-n}}{\Gamma\left(\frac{\alpha}{2}\right)}, \tag{2.5}$$

where α is a complex parameter. Now, from (2.4) and (2.5), we obtain

$$E(x) = -S_{2,c}(x). \tag{2.6}$$

Lemma 2.1. *$R_{\alpha,c}(x)$ and $S_{\alpha,c}(x)$ are homogeneous distributions of order $(\alpha - n)$. In particular, it is a tempered distribution.*

Proof. We need to show that $R_\alpha(x)$ satisfies the Euler equation

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} R_{\alpha,c}(x) = (\alpha - n)R_{\alpha,c}(x).$$

Now

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} R_{\alpha,c}(x)$$

$$= \frac{1}{K_n(\alpha)}\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}(c^2(x_1^2 + \cdots + x_p^2) - x_{p+1}^2 - \cdots - x_{p+q}^2)^{\frac{\alpha-n}{2}}$$

$$= \frac{1}{K_n(\alpha)}(\alpha - n)(c^2(x_1^2 + \cdots + x_p^2) - x_{p+1}^2 - \cdots - x_{p+q}^2)^{\frac{\alpha-n-2}{2}}$$

$$\times (c^2(x_1^2 + \cdots + x_p^2) - x_{p+1}^2 - \cdots - x_{p+q}^2)$$

$$= \frac{1}{K_n(\alpha)}(\alpha - n)(c^2(x_1^2 + \cdots + x_p^2) - x_{p+1}^2 - \cdots - x_{p+q}^2)^{\frac{\alpha-n}{2}}$$

$$= \frac{(\alpha - n)V^{\frac{\alpha-n}{2}}}{K_n(\alpha)}$$

$$= (\alpha - n)R_{\alpha,c}(x).$$

Hence $R_{\alpha, c}(x)$ is a homogeneous distribution of order $(\alpha - n)$ as required and similarly $S_{\alpha, c}(x)$ is also homogeneous distribution of order $(\alpha - n)$. □

Lemma 2.2. *$R_{\alpha, c}(x)$ and $S_{\alpha, c}(x)$ are the tempered distributions.*

Proof. The proof of this lemma is given by Donoghue [2, pp. 154-155] which is stated that every homogeneous distribution is a tempered distribution. □

Lemma 2.3 (The convolutions of tempered distributions)**.**

$$S_{\alpha, c}(x) * S_{\beta, c}(x) = S_{\alpha+\beta, c}(x). \tag{2.7}$$

Proof. The proof of this lemma is also given by Donoghue [2, pp. 156-159]. Now, from (2.6) and (2.7) with $\alpha = \beta = 2$, we obtain

$$\begin{aligned} E(x) * E(x) &= (-S_{2, c}(x)) * (-S_{2, c}(x)) \\ &= (-1)^2 S_{2+2, c}(x) \\ &= S_{4, c}(x). \end{aligned}$$

By induction, we obtain

$$\underbrace{E(x) * E(x) * \cdots * E(x)}_{k\text{-times}} = (-1)^k S_{2k, c}(x). \tag{2.8}$$

□

Lemma 2.4. *Given the equation $\Delta_c^k u(x) = \delta$, where Δ_c^k is the operator related to the Laplace operator iterated k-times defined by*

$$\Delta_c^k = \left[\frac{1}{c^2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right) + \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)\right]^k$$

and $x \in \mathbb{R}^n$, then $u(x) = (-1)^k S_{2k, c}(x)$ is an elementary solution of the operator Δ_c^k, where $(-1)^k S_{2k, c}(x)$ is defined by (2.8).

Proof. Now $\Delta_c^k u(x) = \delta$ can be written in the form $\Delta_c^k \delta * u(x) = \delta$. Convolving both sides by the function $E(x)$ defined by (2.4), we obtain

$$(E(x) * \Delta_c^k \delta) * u(x) = E(x) * \delta = E(x)$$

and

$$(\Delta_c E(x) * \Delta_c^{k-1} \delta) * u(x) = E(x).$$

We have

$$(\delta * \Delta_c^{k-1} \delta) * u(x) = E(x).$$

Since $\Delta_c E(x) = \delta$,

$$(\Delta_c^{k-1} \delta) * u(x) = E(x).$$

By keeping on convolving $E(x)$, $k-1$-times, we obtain

$$\delta * u(x) = \underbrace{E(x) * E(x) * \cdots * E(x)}_{k\text{-times}}.$$

It follows that $u(x) = (-1)^k S_{2k,c}(x)$ by (2.8) as required. Before going to the proofs of theorems, we need to define the convolution of $(-1)^k S_{2k,c}(x)$ with $R_{2k,c}(x)$ defined by (2.2) with $\alpha = 2k$ and $k = 0, 1, 2, \ldots$. Now, for the case $2k \geq n$, we obtain that $(-1)^k S_{2k,c}(x)$ and $R_{2k,c}(x)$ are analytic functions that are the ordinary functions, thus the convolution

$$(-1)^k S_{2k,c}(x) * R_{2k,c}(x) \tag{2.9}$$

exists. Now, for the case $2k < n$, by Lemma 2.2 with $\alpha = 2k$, we obtain $(-1)^k S_{2k,c}(x)$ and $R_{2k,c}(x)$ are tempered distributions.

Let K be a compact set and $K \subset \overline{\Gamma}_+$, where $\overline{\Gamma}_+$ is defined as the beginning. Choose the support of $R_{2k,c}(x)$ equal to K, then $\operatorname{supp} R_{2k,c}(x)$ is compact (close and bounded). Then, by Donoghue [2, pp. 152-153], the convolution

$$(-1)^k S_{2k,c}(x) * R_{2k,c}(x) \tag{2.10}$$

exists and is a tempered distribution. □

3. Main Results

Theorem 3.1. *Given the equation* $\lozenge_c^k u(x) = \delta$, *where* $\lozenge_c^k$ *is the operator related to the diamond operator iterated k-times defined by* (1.1) *and* $x \in \mathbb{R}^n$, *then* $u(x) = (-1)^k S_{2k,c}(x) * R_{2k,c}(x)$ *defined by* (2.9) *and* (2.10) *is a unique elementary solution of the operator* $\lozenge_c^k$.

Proof. Now $\lozenge_c^k u(x) = \delta$ can be written as

$$\lozenge_c^k u(x) = \Box_c^k \Delta_c^k u(x) = \delta.$$

By Trione [8], Kananthai [4] and Tellez [7, pp. 147-149], we have that

$$\Delta_c^k u(x) = R_{2k,c}(x) \tag{3.1}$$

is a unique elementary solution of the operator $\Box_c^k$ for *n* odd integer with *p* odd and *q* even, or for *n* even with *p* and *q* odd integers. Also, we know that

$$\Delta_c^k \delta * u(x) = R_{2k,c}(x). \tag{3.2}$$

Convolution both sides of (3.2) by $(-1)^k S_{2k,c}(x)$, we have

$$[(-1)^k S_{2k,c}(x) * \lozenge_c^k \delta] * u(x) = (-1)^k S_{2k,c}(x) * R_{2k,c}(x)$$

or

$$\Delta_c^k [(-1)^k S_{2k,c}(x)] * u(x) = (-1)^k S_{2k,c}(x) * R_{2k,c}(x).$$

It follows that

$$u(x) = (-1)^k S_{2k,c}(x) * R_{2k,c}(x),$$

by Lemma 2.4. □

Theorem 3.2. *Given the equation*

$$\lozenge_c^r [(-1)^k S_{2k,c}(x) * R_{2k,c}(x)] = (-1)^{k-r} S_{2k-2r,c}(x) * R_{2k-2r,c}(x) \quad \textit{for} \quad 0 < r < k$$

and

$$\lozenge_c^m[(-1)^k S_{2k,c}(x) * R_{2k,c}(x)] = \lozenge_c^{m-k}\delta \quad \textit{for} \quad k \le m.$$

Proof. From Theorem 3.1, $\lozenge_c^k[(-1)^k S_{2k,c}(x) * R_{2k,c}(x)] = \delta.$ Thus

$$\lozenge_c^{k-r}\lozenge_c^r[(-1)^k S_{2k,c}(x) * R_{2k,c}(x)] = \delta$$

or

$$\lozenge_c^{k-r}\delta * \lozenge_c^r[(-1)^k S_{2k,c}(x) * R_{2k,c}(x)] = \delta.$$

Convolving both sides by $(-1)^{k-r} S_{2k-2r,c}(x) * R_{2k-2r,c}(x),$ we obtain

$$\lozenge_c^{k-r}[(-1)^{k-r} S_{2k-2r,c}(x) * R_{2k-2r,c}(x)] * \lozenge_c^r[(-1)^k S_{2k,c}(x) * R_{2k,c}(x)]$$

$$= (-1)^{k-r} S_{2k-2r,c}(x) * R_{2k-2r,c}(x) * \delta$$

or

$$\delta * \lozenge_c^r[(-1)^k S_{2k,c}(x) * R_{2k,c}(x)] = (-1)^{k-r} S_{2k-2r,c}(x) * R_{2k-2r,c}(x)$$

by Theorem 3.1.

It follows that

$$\lozenge_c^r[(-1)^k S_{2k,c}(x) * R_{2k,c}(x)] = (-1)^{k-r} S_{2k-2r,c}(x) * R_{2k-2r,c}(x)$$

for $0 < r < k.$ For $k \le m,$ we have

$$\lozenge_c^m[(-1)^k S_{2k,c}(x) * R_{2k,c}(x)] = \lozenge_c^{m-k}\lozenge_c^k[(-1)^k S_{2k,c}(x) * R_{2k,c}(x)] = \lozenge_c^{m-k}\delta$$

by Theorem 3.1. □

Theorem 3.3. *Given the differential equation*

$$\lozenge_c^k u(x) = \sum_{r=0}^{m} C_r \lozenge_c^r \delta, \tag{3.3}$$

where $\lozenge_c^k$ *is the operator related to the diamond operator iterated k-times and is* *defined by*

$$\Diamond_c^k = \left[\frac{1}{c^4}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^2\right]^k, \quad (3.4)$$

$p + q = n$, *n is odd with p odd and q even or n even with p odd and q odd*, $x \in \mathbb{R}^n$, *C_r is a constant*, *δ is the Dirac-delta distribution and $\Diamond_c^0\delta = \delta$. Then, the type of solution* (3.3) *that depends on the relationship between the values of k and m is as the following cases*:

(1) *if $m < k$ and $m = 0$, then* (3.3) *has the solution*

$$u(x) = C_0(-1)^k S_{2k,c}(x) * R_{2k,c}(x),$$

which is an elementary solution of the operator $\Diamond_c^k$ in Theorem 3.1, *is an ordinary function for $2k \geq n$ and is a tempered distribution for $2k < n$*;

(2) if $0 < m < k$, *then the solution of* (3.3) *is*

$$u(x) = \sum_{r=1}^{m} [(-1)^{k-r} C_r S_{2k-2r,c}(x) * R_{2k-2r,c}(x)]$$

which is an ordinary function for $2k - 2r \geq n$ and a tempered distribution for $2k - 2r < n$;

(3) *if $m \geq k$ and suppose $k \leq m \leq M$, then* (3.3) *has solution*

$$u(x) = \sum_{r=k}^{M} C_r \Diamond_c^{r-k}\delta,$$

which is only the singular distribution.

Proof. (1) For $m = 0$, we have $\Diamond_c^k u(x) = C_0\delta$ and by Theorem 3.1, we obtain $u(x) = C_0(-1)^k S_{2k,c}(x) * R_{2k,c}(x)$. Now $(-1)^k S_{2k,c}(x)$ and $R_{2k,c}(x)$ are the analytic functions for $2k \geq n$ and also $(-1)^k S_{2k,c}(x) * R_{2k,c}(x)$ exists and is an analytic function by (2.9). It follows that $(-1)^k S_{2k,c}(x) * R_{2k,c}(x)$ is an ordinary function for $2k \geq n$. By Lemma 2.2 with $\alpha = 2k$, $(-1)^k S_{2k,c}(x)$ and $R_{2k,c}(x)$ are tempered distributions with $2k < n$ and by (2.10), we obtain that

$$(-1)^k S_{2k,c}(x) * R_{2k,c}(x)$$

exists and is a tempered distribution.

(2) For the case $0 < m < k$, we have

$$\Diamond_c^k u(x) = C_1 \Diamond_c \delta + C_1 \Diamond_c^2 \delta + \cdots + C_m \Diamond_c^m \delta.$$

Convolution both sides by $(-1)^k S_{2k,c}(x) * R_{2k,c}(x)$, we obtain

$$\begin{aligned}\Diamond_c^k [(-1)^k S_{2k,c}(x) * R_{2k,c}(x)] * u(x) = {} & C_1 \Diamond_c [(-1)^k S_{2k,c}(x) * R_{2k,c}(x)] \\ & + C_2 \Diamond_c^2 [(-1)^k S_{2k,c}(x) * R_{2k,c}(x)] \\ & + \cdots + C_m \Diamond_c^m [(-1)^k S_{2k,c}(x) * R_{2k,c}(x)].\end{aligned}$$

By Theorems 3.1 and 3.2, we have

$$\begin{aligned}u(x) = {} & C_1 \Diamond_c [(-1)^{k-1} S_{2k-2,c}(x) * R_{2k-2,c}(x)] \\ & + C_2 \Diamond_c^2 [(-1)^{k-2} S_{2k-4,c}(x) * R_{2k-4,c}(x)] \\ & + \cdots + C_m \Diamond_c^m [(-1)^{k-m} S_{2k-2m,c}(x) * R_{2k-2m,c}(x)] \\ = {} & \sum_{r=1}^{m} (-1)^{k-r} C_r S_{2k-2r,c}(x) * R_{2k-2r,c}(x).\end{aligned}$$

Similarly, as in Case 1, $u(x)$ is an ordinary function for $2k - 2r \geq n$ and is a tempered distribution for $2k - 2r < n$.

(3) For the case $m \geq k$ and suppose $k \leq m \leq M$. Then we have

$$\Diamond_c^k u(x) = C_k \Diamond_c^k \delta + C_{k+1} \Diamond_c^{k+1} \delta + \cdots + C_M \Diamond_c^M \delta.$$

Convolution both sides by $(-1)^k S_{2k,c}(x) * R_{2k,c}(x)$, we obtain

$$\begin{aligned}\Diamond_c^k [(-1)^k S_{2k,c}(x) * R_{2k,c}(x)] * u(x) = {} & C_k \Diamond_c^k [(-1)^k S_{2k,c}(x) * R_{2k,c}(x)] \\ & + C_{k+1} \Diamond_c^{k+1} [(-1)^k S_{2k,c}(x) * R_{2k,c}(x)] \\ & + \cdots + C_M \Diamond_c^M [(-1)^k S_{2k,c}(x) * R_{2k,c}(x)].\end{aligned}$$

By Theorems 3.1 and 3.2, we finally obtain

$$u(x) = C_k\delta + C_{k+1}\Diamond_c\delta + \cdots + C_M\Diamond_c^{M-k}\delta = \sum_{r=k}^{M} C_r\Diamond_c^{r-k}\delta.$$

This finishes the proof of Theorem 3.3. □

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