THE ANTICAUSAL SOLUTIONS OF THE PARTIAL DIFFERENTIAL OPERATOR $\diamondsuit_m^k$ AND $\diamondsuit_{B,m}^k$ RELATED TO THE DIAMOND OPERATOR AND BESSEL-DIAMOND OPERATOR

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SAKON NAKHON RAJABHAT UNIVERSITY

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CHAPTER I

INTRODUCTION

Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n - dimensional space $\mathbb{R}^n$,

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \qquad p + q = n \tag{1.1.1}$$

where $p + q = n$. Define $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ which designates the interior of the forward cone and $\overline{\Gamma}_+$ designates its closure and the following functions introduce by Nozaki ([16], p.72) that

$$R_\alpha(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)} & \text{if } x \in \Gamma_+ \\ 0 & \text{if } x \notin \Gamma_+, \end{cases} \tag{1.1.2}$$

$R_\alpha(u)$ is called the *ultra-hyperbolic kernel of Marcel Riesz.* Here α is a complex parameter and n the dimension of the space. The constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)} \tag{1.1.3}$$

and p is the number of positive terms of

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \qquad p + q = n$$

and let supp $R_\alpha(x) \subset \overline{\Gamma}_+$. Now $R_\alpha(x)$ is an ordinary function if Re $(\alpha) \geq n$ and is a distribution of α if Re $(\alpha) < n$.

Let $x = (x_1, \ldots, x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$. Consider a nondegenerate quadratic form in n variables of the form

$$P = P(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{1.1.4}$$

where $p + q = n$. The distributions $(P \pm i0)^\lambda$ are defined by

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \{P \pm i\varepsilon |x|^2\}^\lambda$$

where $\varepsilon > 0, |x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2, \lambda \in \mathbb{C}$.

Moreover the distribution $(P \pm i0)^\lambda$ are analytic in λ every where except at $\lambda = -\frac{n}{2} - k, k = 0, 1, \ldots$ where they have simple poles.

Similarly, the distribution $(m^2 + P \pm i0)^\lambda$ is denote by

$$(m^2 + P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \{m^2 + P \pm i\varepsilon |x|^2\}^\lambda$$

where m is a real positive number. ([4], p.289)

Following Trione ([12], p.32) by causal (anticausal) distributions we mean distributions of the form $T(P \pm i0, \lambda), P = P(x), T(P \pm i0, \lambda) = (P \pm i0)^\lambda f(P \pm i0, \lambda), f(z, \lambda)$ an entire function in the variables z, λ .

Let

$$G_\alpha(P \pm i0, m, n) = H_\alpha(m, n)(P \pm i0)^{\frac{1}{2}(\frac{\alpha - n}{2})} K_{(\frac{n-\alpha}{2})}(\sqrt{m^2(P \pm i0)}) \qquad (1.1.5)$$

where m is a real positive real number $\alpha \in \mathbb{C}$, K_ν designates the modified Bessel function of the third kind

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sec \pi\nu}, I_\nu(z) = \sum_{m=0}^{\infty} \frac{(\frac{z}{2})^{2m+r}}{m!\Gamma(m + \nu + 1)}$$

and

$$H_\alpha(m, n) = \frac{2^{\frac{1-(\alpha+n)}{2}} (m^2)^{(\frac{1}{2})(\frac{\alpha-n}{2})} e^{\frac{\pi}{2} qi}}{\pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})}$$

We introduce an auxiliary weight function

$$\lambda_\alpha(P \pm i0, m, n) = e^{iq\frac{\pi}{2}} 2^{\frac{1-(\alpha+n)}{2}} (m^2)^{(\frac{1}{2})(\frac{\alpha-n}{2})} (P \pm i0)^{\frac{(n+\alpha)}{4}} K_{\frac{(n+\alpha)}{2}} \sqrt{m^2(P \pm i0)}$$

that is a causal (anticausal) analoque to the auxiliary weight function introduce by Rubin ([2], p. 1247).

Let us define the n-dimensional ultrahyperbolic Klein-Gordon operator iterated k-times

$$(\Box + m^2)^k = \left[\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} + m^2 \right]^k$$

, where

$$\Box = \left[\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right]. \qquad (1.1.6)$$

The distributional function $G_{2k}(P \pm i0, m, n)$ where n is an integer ≥ 2 and $k = 1, 2, 3, \ldots$ are elementary causal (anticausal) solutions of the ultrahyperbolic Klein-Gordon operator iterated k-times

$$(\Box + m^2)^k G_{2k}(P \pm i0, m, n) = \delta.$$

Gelfand and Shilov [4] have first introduced the elementary solution of the n-Dimensional Classical Diamond Operator, and have defined the distribution $(P \pm i0)^\lambda$ as

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \{P \pm i\varepsilon |x|^2\}^\lambda$$

where $\varepsilon > 0, |x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2, \lambda \in \mathbb{C}$. The distributions $(P \pm i0)^\lambda$ are an important contribution of Gelfand and Shilov.

Moreover the distribution $(P \pm i0)^\lambda$ are analytic in λ every where except at $\lambda = -\frac{n}{2} - k, k = 0, 1, \ldots$ where they have simple poles.

Kananthai [1] has studied the solution of n-dimensional diamond operator and the first introduce diamond operator, defined by

$$\diamondsuit^k = \left[\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}^{\,2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}^{\,2} \right]^k \tag{1.1.7}$$

, where $p+q = n$. Bupasiri [9] has studied the solution of n-dimensional operator related to the diamond operator and defined by

$$\diamondsuit^k = \left[\frac{1}{c^4} \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}^{\,2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}^{\,2} \right]^k \tag{1.1.8}$$

, where $p + q = n$

The operator $\diamondsuit^k$ can be written as $\diamondsuit^k = \triangle^k \Box^k = \Box^k \triangle^k$. Thus, the operator $\diamondsuit_m^k$ can be factorized in the following form

$$\begin{aligned} \diamondsuit_m^k &= \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + \frac{m^2}{2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} - \frac{m^2}{2} \right)^2 \right]^k \\ &= \left[\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} + m^2 \right]^k \left[\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k \end{aligned} \tag{1.1.9}$$

, where $p+q=n$ is the dimension of $\mathbb{R}^n$, k is a nonnegative integer. The Laplace operator and the ultra-hyperbolic Klein Gordon operator are defined by

$$\triangle = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \tag{1.1.10}$$

and

$$\Box + m^2 = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} + m^2 \tag{1.1.11}$$

, where

$$\Box = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \tag{1.1.12}$$

is the ultra-hyperbolic operator. Thus, equation (1.1.9) can be written as

$$\diamondsuit_m^k = \triangle^k (\Box + m^2)^k = (\Box + m^2)^k \triangle^k . \tag{1.1.13}$$

In 1988, Trione [10] studied the elementary solution of the ultra-hyperbolic Klein Gordon operator, which iterated k-times, and is defined by

$$(\Box + m^2)^k = \left[\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} + m^2 \right]^k . \tag{1.1.14}$$

We obtain the elementary solution $W_{2k}(P \pm i0, m)$, defined by

$$W_{2k}(P \pm i0, m) = \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r R_{2k+2r}(P \pm i0), \tag{1.1.15}$$

where $R_{2k+2r}(P \pm i0)$ is defined by (1.1.18).

Therefore, $W_{2k}(P \pm i0, m)$ is the unique elementary retarded $(P \pm i0)^\lambda$-ultrahyperbolic solution of the Klein-Gordon operator, iterated k -times. That is

$$(\Box + m^2)^k W_{2k}(P \pm i0, m) = (\Box + m^2)^k (\Box + m^2)^{-k} \delta = \delta \tag{1.1.16}$$

Putting $k = 1$, the formula (1.1.15) says that $W_2(P \pm i0, m)$ is the unique elementary retarded $(P \pm i0)^\lambda$-ultrahyperbolic solution of the Klein-Gordon operator

Now, the purpose of this work is to find the elementary solution of the operator $\diamondsuit_m^k$, that is

$$\diamondsuit_m^k (P \pm i0) = \delta,$$

where $(P \pm i0)$ is the elementary solution, δ is the Dirac-delta distribution, k is a nonnegative integer and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n - dimensional space $\mathbb{R}^n$,

$$P = P(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{1.1.17}$$

where $p + q = n$. Define $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } P > 0\}$ which designates the interior of the forward cone and $\overline{\Gamma}_+$ designates its closure and the following functions introduce by Nozaki ([16], p.72) that

$$R_\alpha = R_\alpha(P \pm i0) = \begin{cases} \frac{(P \pm i0)^{\frac{\alpha-n}{2}}}{K_n(\alpha)} & \text{if } x \in \Gamma_+ \\ 0 & \text{if } x \notin \Gamma_+, \end{cases} \tag{1.1.18}$$

R_α is called the *ultra-hyperbolic kernel of Marcel Riesz.* Here α is a complex parameter and n the dimension of the space. The constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma \frac{2+\alpha-n}{2} \Gamma \frac{1-\alpha}{2} \Gamma(\alpha)}{\Gamma \frac{2+\alpha-p}{2} \Gamma \frac{p-\alpha}{2}} \tag{1.1.19}$$

and p is the number of positive terms of

$$P = P(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \qquad p+q = n \tag{1.1.20}$$

and let supp $R_\alpha(P \pm i0) \subset \overline{\Gamma}_+$. Now R_α is an ordinary function if Re $(\alpha) \geq n$ and is a distribution of α if Re $(\alpha) < n$.

Now, we define the causal (anticausal) distributions $S_\alpha(P' \pm i0)$ as follows:

$$S_\alpha = S_\alpha(P' \pm i0) \frac{e^{\frac{i\pi\alpha}{2}} e^{\pm i\frac{\pi}{2}q} \Gamma(\frac{n-\alpha}{2})}{2^\alpha \pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})} (P' \pm i0)^{\frac{\alpha-n}{2}} \tag{1.1.21}$$

where $\alpha \in \mathbb{C}$,

$$P' = P'(x) = x_1^2 - x_2^2 - \cdots - x_n^2$$

and q is the number of negative terms of the quadratic form P . The distributional functions S_α are the causal (anticausal) analogues of the elliptic kernel of M. Riesz ([7], pp.16-21), and have analogous properties [13].

CHAPTER II

BASIC CONCEPTS AND PRELIMINARIES

In this chapter, we studied some properties of the test function, the distribution, the gamma function , Causal and anticausal Solution of the Operator $\diamondsuit_m^k$ which will be used in later chapters.

2.1 Test functions

Let $\mathbb{R}^n$ be a real n-dimensional space in which we have a Cartesian system of coordinates such that a point P is denoted by $x = (x_1, x_2, \ldots, x_n)$ and the distance r, of P from the origin, is $r = |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. Let k be an n-tuple of nonnegative integer, $k = (k_1, k_2, \ldots, k_n)$, the so-called *multiindex* of order n; then we define

$$|k| = k_1 + k_2 + \cdots + k_n, \quad x^k = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

and

$$D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}} = \frac{\partial^{k_1+k_2+\cdots+k_n}}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}} = D_1^{k_1} D_2^{k_2} \cdots D_n^{k_n},$$

where $D_j = \partial/\partial x_j, j = 1, 2, \ldots, n$. For the one-dimensional case, D^k reduces to d/dx. Furthermore, if any component of k is zero, the differentiation with respect to the corresponding variable is omitted.

Example 2.1.1. In $\mathbb{R}^3$, with $k = (3, 0, 4)$, we have

$$D^k = \partial^7/\partial x_1^3 \partial x_3^4 = D_1^3 D_3^4.$$

Definition 2.1.2. A function $f(x)$ is *locally integrable* in $\mathbb{R}^n$ if $\int_R |f(x)|dx$ exists for every bounded region R in $\mathbb{R}^n$. A function $f(x)$ is locally integrable on a hypersurface in $\mathbb{R}^n$ if $\int_S |f(x)|dS$ exists for every bounded region S in $\mathbb{R}^{n-1}$.

Definition 2.1.3. The *support* of a function $f(x)$ is the closure of the set of all points x such that $f(x) \neq 0$. We shall denote the support of f by supp f.

Example 2.1.4. For $f(x) = \sin x, x \in \mathbb{R}$, the support of $f(x)$ consists of the whole real line, even though $\sin x$ vanishes at $x = n\pi$.

Definition 2.1.5. ([6]). If supp f is a bounded set, then f is said to have a *compact support.*

We have observed that an operational quantity such as $\delta(x)$ becomes meaningful if it is first multiplied by a sufficiently smooth auxiliary function and then integrated over the entire space. This point of view is also taken as the basis for the definition of an arbitrary generalized function. Accordingly, consider the space D consisting of real-valued functions $\phi(x) = \phi(x_1, x_2, \ldots, x_n)$, such that the following hold:

(1) $\phi(x)$ is an infinitely differentiable function defined at every point of $\mathbb{R}^n$. This means that $D^k\phi$ exists for all multiindices k. Such a function is also called a C^∞ function.

(2) There exists a number A such that $\phi(x)$ vanishes for $r > A$. This means that $\phi(x)$ has a compact support. Then $\phi(x)$ is called a test function.

Example 2.1.6. The support of the function

$$f(x) = \begin{cases} 0, & \text{for } -\infty < x \leq -1 \\ x+1, & \text{for } -1 < x < 0 \\ 1-x, & \text{for } 0 \leq x < 1 \\ 0, & \text{for } 1 \leq x < \infty \end{cases}$$

is $[-1, 1]$, which is compact.

Example 2.1.7. The prototype of a test function belonging to D is

$$\phi(x, a) = \begin{cases} \exp\left(-\frac{a^2}{a^2-r^2}\right), & \text{for } r < a \\ 0, & \text{for } r > a. \end{cases} \tag{2.1.1}$$

Its support is clearly $r \leq a$.

The following properties of the test functions are evident.

(1) If ϕ_1 and ϕ_2 are in D, then so is $c_1\phi_1 + c_2\phi_2$, where c_1 and c_2 are real numbers. Thus D is a linear space.

(2) If $\phi \in D$, then so is $D^k\phi$.

(3) For a C^∞ function $f(x)$ and $\phi \in D, f\phi \in D$.

(4) If $\phi(x_1, x_2, \ldots, x_m)$ is an m-dimensional test function and $\psi(x_{m+1}, x_{m+2}, \ldots, x_n)$ is an $(n-m)$ -dimensional test function, then $\phi\psi$ is an n-dimensional test function in the variables $x_1, x_2, \ldots, x_n$.

Definition 2.1.8. The *Schwartz space* or *space of rapidly decreasing functions* S on $\mathbb{R}^n$ is the function space

$$S(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) \mid \|f\|_{\alpha,\beta} < \infty \forall \alpha, \beta \},$$

where α, β are multi-indices, $C^\infty(\mathbb{R}^n)$ is the set of smooth functions from $\mathbb{R}^n$ to $\mathbb{C}$, and

$$\|f\|_{\alpha,\beta} = \|x^\alpha D^\beta f\|_\infty.$$

Here, $\|\cdot\|_\infty$ is the supremum norm, and we use multi-index notation.

Example 2.1.9. If i is a multi-index, and is a positive real number, then

$$x^i e^{-ax^2} \in S(\mathbb{R}).$$

Any smooth function f with compact support is in S. This is clear since any derivative of f is continuous, so $(x^\alpha D^\beta)f$ has a maximum in $\mathbb{R}^n$.

Definition 2.1.10. A sequence $\{\phi_m\}, m = 1, 2, \ldots,$ where $\phi_m \in D$, converges to ϕ_0 if the following two conditions are satisfied:

(1) All ϕ_m as well as ϕ_0 vanish outside a common region.

(2) $D^k\phi_m \to D^k\phi_0$ uniformly over $\mathbb{R}^n$ as $m \to \infty$ for all multiindices k.

It is not difficult to show that $\phi_0 \in D$ and hence that D is closed (or is complete) with respect to this definition of convergence. For the special case $\phi_0 = 0$, the sequence $\{\phi_m\}$ is called a null sequence.

Example 2.1.11. The sequence

$$\{(1/m)\phi(x,a)\}, \tag{2.1.2}$$

where $\phi(x,a)$ is defined by (2.1.1), is a null sequence. However, the sequence $(1/m)\phi(x/m,a)$ is not a convergent sequence, because the support of the function $\phi(x/m,a)$ is the sphere with radius ma, which is unique for each m.

In addition to the space D of test functions, we shall use certain subspaces of D. For a region R in $\mathbb{R}^n$, the space D_R contains those test functions whose support lies in R, that is,

$$D_R \equiv \{\phi : \phi \in D, \qquad \text{supp}\ \phi \subset \mathbb{R}\}. \tag{2.1.3}$$

It is clearly a linear subspace of D.

Example 2.1.12. D_x and D_y are two one-dimensional subspaces of test functions $\phi(x)$ and $\phi(y)$ and are contained in D_{xy}, which is the space of test functions $\phi(x,y)$ in $\mathbb{R}^2$. The convergence in D_R is defined in the same manner as that in the space D.

2.2 Distributions

Definition 2.2.1. A linear functional t on the space D of test functions is an operation (or a rule) by which we assign to every test function $\phi(x)$ a real number denoted $\langle t, \phi\rangle$, such that

$$\langle t, c_1\phi_1 + c_2\phi_2\rangle = c_1 \langle t, \phi_1\rangle + c_2 \langle t, \phi_2\rangle \tag{2.2.1}$$

for arbitrary test functions ϕ_1 and ϕ_2 and real numbers c_1 and c_2.

Definition 2.2.2. A linear functional on D is *continuous* if and only if the sequence of numbers $\langle t, \phi_m\rangle$ converges to $\langle t, \phi\rangle$ when the sequence of test functions $\{\phi_m\}$ converges to the test function ϕ. Thus

$$\lim_{m\to\infty} \langle t, \phi_m\rangle = \left\langle t, \lim_{m\to\infty} \phi_m \right\rangle.$$

In physical problems, one often encounters idealized concepts such as a force concentrated at a point ξ or an impulsive force that acts instantaneously. These forces are described by the Dirac-delta function $\delta(x-\xi)$, which has several significant properties:

$$\delta(x-\xi) = 0, x \neq \xi, \tag{2.2.2}$$

$$\int_a^b \delta(x-\xi)dx = \begin{cases} 0, & \text{for } a, b < \xi \text{ or } \xi < a, b \\ 1, & \text{for } a \leq \xi \leq b, \end{cases} \tag{2.2.3}$$

and

$$\int_{-\infty}^{\infty} \delta(x-\xi)dx = 1. \tag{2.2.4}$$

Equation (2.2.4) is a special case of the general formula

$$\int_{-\infty}^{\infty} \delta(x-\xi)f(x)dx = f(\xi), \tag{2.2.5}$$

where $f(x)$ is a sufficiently smooth function. Relation (2.2.5) is called the *sifting property* or the *reproducing property* of the delta function, and (2.2.4) is obtained from it by putting $f(x) = 1$.

We now have all the tools for defining the concept of distributions.

Definition 2.2.3. A continuous linear functional on the space D of test functions is called a *distribution*.

Example 2.2.4. The Heaviside distribution in $\mathbb{R}^n$ is $\langle H_R, \phi\rangle = \int_R \phi(x)dx$, where

$$H_R(x) = \begin{cases} 1 & \text{for } x \in R \\ 0 & \text{for } x \notin R. \end{cases} \tag{2.2.6}$$

For $\mathbb{R}$, (2.2.6) becomes

$$\langle H, \phi\rangle = \int_0^{\infty} \phi(x)dx. \tag{2.2.7}$$

Example 2.2.5. The Dirac delta distribution in $\mathbb{R}^n$ is

$$\langle \delta(x-\xi), \phi(x)\rangle = \phi(\xi) \tag{2.2.8}$$

for ξ is a fixed point in $\mathbb{R}^n$. Linearity of this functional follows from the relation

$$\langle \delta, c_1\phi_1(x) + c_2\phi_2(x)\rangle = c_1\phi_1(\xi) + c_2\phi_2(\xi) = c_1 \langle \delta, \phi_1\rangle + c_2 \langle \delta, \phi_2\rangle, \tag{2.2.9}$$

where c_1 and c_2 are arbitrary real constants.

Definition 2.2.6. A distribution E is said to be an *elementary solution* for the differential operator L if

$$LE = \delta.$$

Example 2.2.7. The function $R_{2k}(u)$ is the elementary solution of the operator $\Box^k$, where $\Box^k$ is defined by (1.1.6) and $R_{2k}(u)$ is defined by (1.1.18) with $\alpha = 2k$. That is, $\Box^k R_{2k}(u) = \delta$, see([5], p.147)

2.3 Gamma functions

Definition 2.3.1. The gamma function is denoted by Γ and is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \tag{2.3.1}$$

where z is a complex number with $Re\ z > 0$

Example 2.3.2. Show that $\Gamma(1) = 1$.

Proof. By definition 2.3.1, we obtain

$$\begin{aligned}\Gamma(1) &= \int_0^\infty e^{-t} dt \\ &= \lim_{a\to\infty} (-e^{-t}|_0^a) \\ &= 1.\end{aligned}$$

□

Proposition 2.3.3. ([3]) Let z be a complex number. Then

(1) $z\Gamma(z) = \Gamma(z+1), \quad z \neq 0, -1, -2, \ldots$

(2) $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}, \quad z \neq 0, \pm 1, \pm 2, \ldots$

(2) $\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z), \quad z \neq 0, -1, -2, \ldots$

2.4 Properties of the convolution of distribution

Definition 2.4.1. The convolution $f * g$ of two functions $f(t)$ and $g(t)$, both in $\mathbb{R}^n$, is defined as

$$f * g = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau. \tag{2.4.1}$$

Example 2.4.2. Let

$$f(t) = \begin{cases} l^{-t}, & \text{for } t \geq 0 \\ 0, & \text{for } t \geq 0 \end{cases}$$

and

$$g(t) = \begin{cases} \sin t, & \text{for } 0 \leq t \leq \frac{\pi}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$f * g = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau,$$

$$f * g = \begin{cases} \int_0^t l^{-t} \sin(t-\tau)d\tau, & \text{for } 0 < t < \frac{\pi}{2} \\ \int_{t-\frac{\pi}{2}}^t l^{-t} \sin(t-\tau)d\tau, & \text{for } t \geq \frac{\pi}{2} \\ 0, & \text{for } t < 0. \end{cases}$$

Properties of the Convolution of Distributions

Property 1. Commutativity.

$$s * t = t * s \tag{2.4.2}$$

Property 2. Associativity.

$$(s * t) * u = s * (t * u) \tag{2.4.3}$$

if the supports of the two of these three distributions are bounded or if the supports of all three distributions are bounded on the same side.

Proposition 2.4.3. ([6]). If the convolution $s * t$ exists, then the convolutions $(D^k s) * t$ and $s * (D^k t)$ exist, and

$$(D^k s) * t = D^k(s * t) = s * (D^k t). \tag{2.4.4}$$

If L is a differential operator with constant coefficients, we find from (2.4.4) that

$$(Ls) * t = L(s * t) = s * (Lt). \tag{2.4.5}$$

2.5 Causal (anticausal) distributions

Let $x = (x_1, \ldots, x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$. Consider a nondegenerate quadratic form in n variables of the form

$$P = P(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{2.5.1}$$

where $p + q = n$. The distributions $(P \pm i0)^\lambda$ are defined by

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \{P \pm i\varepsilon |x|^2\}^\lambda$$

where $\varepsilon > 0, |x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2, \lambda \in \mathbb{C}$.

Moreover the distribution $(P \pm i0)^\lambda$ are analytic in λ every where except at $\lambda = -\frac{n}{2} - k, k = 0, 1, \ldots$ where they have simple poles. Similarly, the distribution $(m^2 + P \pm i0)^\lambda$ is denote by

$$(m^2 + P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \{m^2 + P \pm i\varepsilon |x|^2\}^\lambda$$

where m is a real positive number. ([4], p.289)

Following Trione ([12], p.32) by causal (anticausal) distributions we mean distributions of the form $T(P \pm i0, \lambda), P = P(x), T(P \pm i0, \lambda) = (P \pm i0)^\lambda f(P \pm i0, \lambda), f(z, \lambda)$ an entire function in the variables z, λ .

Let

$$G_\alpha(P \pm i0, m, n) = H_\alpha(m, n)(P \pm i0)^{\frac{1}{2}(\frac{\alpha - n}{2})} K_{(\frac{n-\alpha}{2})}(\sqrt{m^2(P \pm i0)}) \tag{2.5.2}$$

where m is a real positive real number $\alpha \in \mathbb{C}$, K_ν designates the modified Bessel function of the third kind

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sec \pi\nu}, I_\nu(z) = \sum_{m=0}^{\infty} \frac{(\frac{z}{2})^{2m+r}}{m!\Gamma(m + \nu + 1)}$$

and

$$H_\alpha(m, n) = \frac{2^{\frac{1-(\alpha+n)}{2}} (m^2)^{(\frac{1}{2})(\frac{\alpha-n}{2})} e^{\frac{\pi}{2} qi}}{\pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})}$$

We introduce an auxiliary weight function

$$\lambda_\alpha(P \pm i0, m, n) = e^{iq\frac{\pi}{2}} 2^{\frac{1-(\alpha+n)}{2}} (m^2)^{(\frac{1}{2})(\frac{\alpha-n}{2})} (P \pm i0)^{\frac{(n+\alpha)}{4}} K_{\frac{(n+\alpha)}{2}} \sqrt{m^2(P \pm i0)}$$

that is a causal (anticausal) analoque to the auxiliary weight function introduce by Rubin ([2], p. 1247).

Let us define the n-dimensional ultrahyperbolic Klein-Gordon operator iterated k-times

$$(\Box + m^2)^k = \left[\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right]^k$$

, The distributional function $G_{2k}(P \pm i0, m, n)$ where n is an integer ≥ 2 and $k = 1, 2, 3, \ldots$ are elementary causal (anticausal) solutions of the ultrahyperbolic Klein-Gordon operator iterated k-times

$$(\Box + m^2)^k G_{2k}(P \pm i0, m, n) = \delta.$$

CHAPTER III

CAUSAL AND ANTICAUSAL SOLUTION OF THE OPERATOR $\diamondsuit_m^k$

In this chapter, we study the causal and anticausal solution of the operator $\diamondsuit_m^k$. Moreover, such a solution is unique.

3.1 Main results

Lemma 3.1.1. *S_α and W_β are tempered distributions.*

Proof. By following Kananthai ([1], p.31), we know that Donoghue ([15], pp.154-155) establishes that every homogeneous distribution is a tempered distribution and see [11].

Lemma 3.1.2. *The distributions $S_{2k} = S_{2k}(P' \pm i0, n), 2k \neq n + 2r$, $r = 0, 1, \ldots,$ are the causal (anticausal) elementary solutions of the homogeneous ultrahyperbolic operator iterated k -times; equivalently, the following formula is valid*

$$\Box^k S_{2k}(P' \pm i0, n) = \delta \tag{3.1.1}$$

here $\Box^k$ is defined by (1.1.12) and $S_{2k}(P' \pm i0, n)$ by the formula (1.1.21).

The proof of this Lemma is given in [14].

Theorem 3.1.3. *The convolution distributional functions*

$$(-1)^k S_{2k}(P' \pm i0) * W_{2k}(P \pm i0, m), \tag{3.1.2}$$

which are defined by (1.1.21) and (1.1.15), respectively, where $k = 1, 2, \ldots, n$ integer ≥ 2; are the elementary solution of the $\diamondsuit_m^k(P \pm i0) = \delta$, where $\diamondsuit_m^k$ is the causal (anticausal) of the operator related to the diamond operator iterated k-times defined by (1.1.9):

$$\diamondsuit_m^k(P \pm i0) = \delta. \tag{3.1.3}$$

Proof. By (1.1.13), equation (3.1.3) can be written as

$$\diamondsuit_m^k(P \pm i0) = \triangle^k(\Box + m^2)^k(P \pm i0) = \delta. \tag{3.1.4}$$

Now convolving both sides of (3.1.4) by $(-1)^k S_{2k}(P' \pm i0) * W_{2k}(P \pm i0, m)$, we obtain

$$\begin{aligned}\triangle^k((-1)^k S_{2k}(P' \pm i0)) * (\Box + m^2)^k(W_{2k}(P \pm i0, m)) * (P \pm i0)\\ = (-1)^k S_{2k}(P' \pm i0) * W_{2k}(P \pm i0, m) * \delta.\end{aligned}$$

By Lemma 3.1.2 and equation (1.1.15), we obtain (3.1.2) as required, we call the solution $(P \pm i0)$ in (3.1.3) the elementary solution of the operator $\diamondsuit_m^k$, we denote the elementary solution

$$(P \pm i0) = (-1)^k S_{2k}(P' \pm i0) * W_{2k}(P \pm i0, m).$$

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CHAPTER II

BASIC CONCEPTS AND PRELIMINARIES

In this chapter, we studied some properties of the test function, the distribution, the gamma function , Causal and anticausal Solution of the Operator $\diamondsuit_m^k$ which will be used in later chapters.

2.1 Causal and anticausal Solution of the Operator $\diamondsuit_m^k$